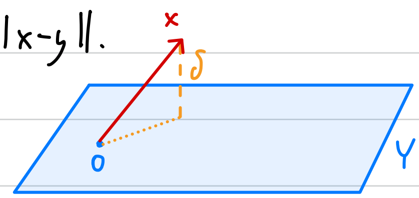
Math 565: Functional Analysis
Lecture 11

Main Hahn-Banach corollaries. Let X be a (complex) normed vector space.

(a) For each closed subspace $Y \subseteq X$ and $x \in X \setminus Y$, $\exists\, f \in X^*$ with $f|_Y = 0$, $f(x) \neq 0$.
In fact, we can ensure $\|f\| = 1$ and $f(x) = \delta := \operatorname{dist}(x, Y) := \inf_{y \in Y} \|x - y\|$.



(b) If $0 \neq x \in X$ then $\exists\, f \in X^*$ s.t. $\|f\| = 1$ and $f(x) = \|x\|$.

(c) Bounded linear functionals separate points of X, i.e. for $x \neq y$ in X, there is $f \in X^*$ with $f(x) \neq f(y)$.

(d) For each $x \in X$, define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) := f(x)$. Then $\|\hat{x}\| = \|x\|$, so the map $x \mapsto \hat{x} : X \to X^{**}$ is an isometry.

Proof. (a) Define $f : Y + \mathbb{C}x \to \mathbb{C}$ by $f(y + \lambda x) = \lambda \cdot \delta$. This is a linear functional with $\|f\| = 1$ because $|f(y+\lambda x)| = |\lambda| \cdot \delta \leq |\lambda| \, \|\tfrac{1}{\lambda} y + x\| = \|y + \lambda x\|$ and $\forall \varepsilon > 0$ we can choose $y \in Y$ so that $0 < \|y + x\| \leq \frac{\delta}{1-\varepsilon}$ (†) hence $z := \frac{1}{\|y+x\|}(y+x)$ has norm 1 and

$$|f(z)| = \frac{1}{\|y+x\|} f(y+x) = \frac{\delta}{\|y+x\|} \overset{(\dagger)}{\geq} \frac{\delta(1-\varepsilon)}{\delta} = 1 - \varepsilon.$$

Now extend f to $\tilde{f} \in X^*$ with $\|\tilde{f}\| = \|f\| = 1$ by the complex Hahn-Banach, getting the desired functional. $\boxed{a}$

(b) This is just (a) with $Y = \{0\}$.

(c) Apply (b) to $x - y$.

(d) Clearly $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\| = \|x\| \cdot \|f\|$, so $\|\hat{x}\| \leq \|x\|$. For equality, apply (b) to x

to get $f \in X^*$ with $\|f\|=1$ and $f(x)=\|x\|$, so $|\hat{x}(f)| = |f(x)| = \|x\|$, hence $\|\hat{x}\| \geq \|x\|$. QED

We denote by $\hat{X}$ the image of X under the isometry $x \mapsto \hat{x} : X \to X^{**}$. Since X^{**} is a Banach space (regardless of whether X is), hence $\hat{X}$ is closed in $X^{**} \iff X$ is Banach. If X is not a Banach space, $\overline{\hat{X}}$ in X^{**} is a completion of X into a Banach space.

We call X reflexive if $\hat{X} = X^{**}$. In particular, reflexive spaces must be Banach spaces. Note that $X \cong X^{**}$ does not imply reflexivity in general because X being reflexive means that specifically the natural map $x \mapsto \hat{x} : X \to X^{**}$ is an isometric isomorphism.

Examples. (a) All finite dimensional normed vector spaces (i.e. $\mathbb{C}^n$) are reflexive (in particular Banach). This is proved in basic linear algebra.

(b) For $1<p<\infty$ and any measure space (X,μ), $L^p(X,\mu)$ is reflexive because

$$(L^p)^{**} \cong (L^q)^* \cong L^p,$$

and it is the natural map $f \mapsto \hat{f} : L^p \to (L^p)^{**}$ that witnesses the isomet. isomorphism $L^p \cong (L^p)^{**}$.

(c) $c_0 := C_0(\mathbb{N})$, i.e. the space of sequences of $\mathbb{C}$ converging to 0, is not reflexive. In fact, $(c_0)^* = \ell^1$ so $(c_0)^{**} = \ell^\infty$, and $c_0 \ncong \ell^\infty$.

Proof. By the Reisz representation theorem, $c_0^* =$ the space $RM(\mathbb{N})$ of all complex Radon measures on $\mathbb{N}$. But $\mathbb{N}$ is discrete, so $\mathcal{B}(\mathbb{N}) = \mathcal{P}(\mathbb{N})$, hence $RM(\mathbb{N}) = \ell^1(\mathbb{N})$. But $(\ell^1)^* \cong \ell^\infty$ and if we identify $(\ell^1)^*$ with ℓ^∞, the natural map $x \mapsto \hat{x} : c_0 \to (c_0)^{**}$ turns into the inclusion $x \mapsto x : c_0 \to \ell^\infty$, which is clearly not surjective, so c_0 is not reflexive. Moreover, $c_0 \ncong \ell^\infty$ by any other map because ℓ^∞ is not separable, while c_0 is: indeed, $c_c^{\mathbb{Q}} := \{x \in (\mathbb{Q}+i\mathbb{Q})^{\mathbb{N}} : x \in C_c$, i.e. has only finitely many nonzero entries$\}$ is ctbl and dense in c_0. □

(d) $\ell^1 := \ell^1(\mathbb{N})$ is not reflexive. In fact, $(\ell^1)^* = \ell^\infty$ and $\ell^1 \subsetneq (\ell^\infty)^*$.

Proof. We have shown $(\ell^1)^{**} = (\ell^\infty)^*$ and $(\ell^\infty)^*$ isn't separable (otherwise, ℓ^∞ would be separable, by HW), hence isn't isomorphic to ℓ^1; in particular, $\ell^1 \subsetneq (\ell^\infty)^*$. □

Geometric Hahn-Banach.

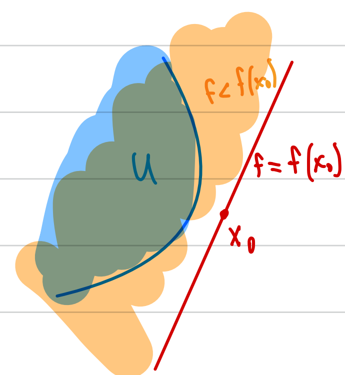
Def. In a (real) vector space X, a set $U \subseteq X$ is called **convex** if it is closed under convex combinations, i.e. $\alpha x + (1-\alpha) y \in U$ whenever $x, y \in U$ and $\alpha \in [0,1]$.

Let X be a normed vector space an $U \subseteq X$ a convex open set with $0 \in U$. Define the **Minkowski functional** $p: X \to \mathbb{R}$ of U (also called the **gauge** of U) by

$$p(x) := \inf \{\alpha > 0 : \alpha^{-1} x \in U\} \geq 0.$$

Lemma 1. If $U \subseteq X$ is convex open and $0 \in U$, then p is a nonnegative sublinear functional (but not a seminorm), which is bounded (i.e. $\exists\, C \geq 0$ s.t. $|p(x)| \leq C\|x\|$ $\forall x \in X$) and $U = \{p < 1\}$.

Lemma 2. Let $U \subseteq X$ be a convex open set and $x_0 \in X \setminus U$. Then $\exists\, f \in X^*$ with $f|_U < f(x_0)$.

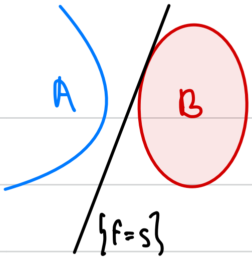


Def. A hyperplane H in a real normed vector space X is a set of the form $H = \{f = c\}$ for some $f \in X^*$ and $c \in \mathbb{R}$.

Remark. The word "hyper" refers to this set being a coset of co-dim 1 subspace $\ker f$. Co-dimension refers to the dimension of the quotient $X/\ker f$.

Thus Lemma 2 says that a convex open set can be separated from a point by a hyperplane. We can boost this from a point to any convex set.

Geometric Hahn-Banach. Let X be a real normed vector space and $A, B \subseteq X$ be disjoint convex sets. If A is open, then A and B are separated by a hyperplane, i.e. $\exists f \in X^*$ and a real $s \in \mathbb{R}$ such that $f|_A < s$ and $f|_B \geq s$.



Proof. Let $U := A - B := \{a - b : a \in A \text{ and } b \in B\}$, hence U is convex and open since $U = \bigcup_{b \in B} (A - b)$ and $A - b$ is open.
Because $A \cap B = \emptyset$, $0 \notin U$ hence Lemma 2 gives an $f \in X^*$ with $f|_U < f(0) = 0$, i.e. $\forall\, a - b \in U$, $f(a-b) < 0$ hence $f(a) < f(b)$. Thus:

$$\sup_{a \in A} f(a) \leq \inf_{b \in B} f(b),$$

so there is a real s in between, hence $f|_A \leq s$ and $f|_B \geq s$. But because A is open and contained in the closed set $\{f \leq s\}$, it does not intersect the boundary of this closed set, namely, $\{f = s\}$. Hence, $f|_A < s$. QED

Application of Hahn-Banach: Banach limits (aka means on $\mathbb{N}$).

Let $c := c(\mathbb{N})$ be the subspace of $\ell^\infty := \ell^\infty(\mathbb{N})$ of all convergent sequences in $\mathbb{C}$.

Note. Letting $\mathbb{1} := \mathbb{1}_{\mathbb{N}} = (1, 1, 1, \ldots)$, we see that $c = \mathbb{C} \cdot \mathbb{1} + c_0$.

Obs. c_0 is a closed subspace of c (in the uniform norm) and c is a closed subspace of ℓ^∞ (also in the uniform norm).

Prop. The map $L : c \to \mathbb{C}$ by $L(x) := \lim_{n \to \infty} x(n)$ is a linear functional with $\|L\| = 1$ satisfying $L(\mathbb{1}) = 1$ and L is shift invariant, i.e. $L \circ s = L$, where $s : \ell^\infty \to \ell^\infty$ by $(x_n)_{n \in \mathbb{N}} \mapsto (x_{n+1})_{n \in \mathbb{N}}$. Also, L is a positive linear functional, i.e. if $x \geq 0$, then $L(x) \geq 0$. Conversely, these conditions uniquely determine L.

Proof. The forward direction is immediate, while the converse is HW. $\square$

The following theorem allows for taking limits of bounded sequences that do not converge:

Existence of Banach limits. $\exists\, \tilde{L} \in (\ell^\infty)^*$ such that

(i) $\tilde{L} \circ S = \tilde{L}$ (shift-invariant);

(ii) $\|\tilde{L}\| = 1$;

(iii) $\tilde{L}|_c = L$;

(iv) $\tilde{L}$ is positive, i.e. $x \geq 0 \Rightarrow \tilde{L}x \geq 0$. In particular, if $x \in \ell^\infty_{\mathbb{R}}$, then $Lx \in \mathbb{R}$.

Proof. The "in particular" part in (iv) follows from the nonnegativity of $\tilde{L}$ because if $x \in \ell^\infty_{\mathbb{R}}$, then $x = x_+ - x_-$ with $x_+, x_- \geq 0$, hence $\tilde{L}(x) = \tilde{L}(x_+) - \tilde{L}(x_-) = \text{nonneg} - \text{nonneg} \in \mathbb{R}$.

We first define $\tilde{L}$ on $\ell^\infty_{\mathbb{R}}$, and then uniquely extend (by linearity) to $\ell^\infty_{\mathbb{C}}$.